RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. SECOND SEMESTER EXAMINATION, MAY 2018

FIRST YEAR [BATCH 2017-20]

PHYSICS (Honours)

Paper : II

Date : 19/05/2018

Time : 11 am – 3 pm

Full Marks : 100

[Use a separate Answer Book for each group]

Group - A

(*Answer* *<u>any three</u>* *questions*) [3×10]

1. a) What do you mean by dimension of a vector space? Find out the dimension of the vector space spanned by the columns of

$$\begin{pmatrix} 2 & 3 & -1 \\ -1 & 4 & -16 \\ 0 & -8 & 24 \end{pmatrix}.$$ [1+2]

b) Given $A \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B \equiv \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ How the eigenvalues and eigenvectors of these two matrices are related? [3]

c) A matrix A is given by $\begin{pmatrix} 2 & 4 \\ 5 & 6 \end{pmatrix}$ in the standard basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find A with respect to the basis $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$. [4]

2. a) Consider the coordinate system $OX_1 - X_2 - X_3$. In that particular basis; two vectors $\vec{r}$ and $\vec{R}$ are represented. Those two vectors are connected by a square matrix A; $\vec{R} = A\vec{r}$.

Transform it to a new system $OX_1' - X_2' - X_3'$ (a new basis) and the relation becomes $\vec{R}' = A'\vec{r}'$. Components of a vector in the prime and unprime basis are connected by equation $\vec{r} = S\vec{r}'$ and $\vec{R} = S\vec{R}'$ where S is a non-singular matrix.

i) What is the relation between A and A'? Deduce it. [3]

ii) Show that the relation $A\vec{R} = B\vec{r}$ is invariant under a similarity transformation. [2]

b) Find the eigenvalues and eigenvectors of a 6×6 identity matrix. [1]

c) Show that the transpose of the product matrix AB is $B^T A^T$. [2]

d) Can you diagonalize the matrix $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$? Justify your answer. [2]

3. a) i) Show that the tensor $T_{jkmn} = \epsilon_{ijk}\epsilon_{imn}$ is antisymmetric with respect to the indices j and k and also with respect to m and n, where $\in$'s are Levi-civita. [2]

ii) If A_{ik} is an antisymmetric tensor, prove that $(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{kj})A_{ik} = 0$. [1]

b) Establish the relationship between the Gamma function and Beta function i.e., $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, where the symbols have their usual meanings. [3]

c) Show that : $\int_{-\infty}^{\infty} dx e^{-|x|}\delta(\sin x) = \cot h\dfrac{\pi}{2}$

[Hint : use $\delta(g(x)) = \sum_i \dfrac{\delta(x - x_i)}{|g'(x_i)|}$ where the sum runs over all the real zeros of g(x)] [4]

4. a) Given a set of functions $\xi_n(x) = x^n$, n = 0, 1, 2, ... $x \in (-\infty, +\infty)$.

i) Show that these functions are not square-integrable functions. [1]

ii) Now an appropriate weight (damping factor) is introduced with each i.e $\psi_n(x) = \xi_n e^{-\frac{1}{2}x^2}$, n = 0, 1, 2 ... Now show that these set of functions are square-integrable. [3]

b) Show that $G(x,h) = \exp\left[\frac{x}{2}\left(h - \frac{1}{h}\right)\right]$ is the generating function for the Bessel functions. Using the generating function for integer n prove the recursion relation $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$. [3+3]

5. a) Determine the critical points of the function $f(x,y) = x^3 + y^3 - 3xy$. Find whether the critical points are a relative maximum, relative minimum or a saddle point. [2]

b) A homogeneous rectangular membrane $(0 \le x \le a;\ 0 \le y \le b)$ fixed along the contour vibrates. Find the free vibration law of the membrane, if the initial conditions are

$u(x,y,0) = xy(a-x)(b-y)$ and $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$. [8]

Group - B

(*Answer <u>any two</u> questions*) [2×10]

6. a) Write down the equations of motion of a system of N mass-points m_i (i = 1, 2, ..., N) moving under the action of applied forces $\vec{F}_i$ and pair-wise internal forces $\vec{f}_{ij}$. If the internal forces are central forces obeying Newton's third law, prove the following :

i) For translational motion, $\frac{d}{dt}(M\dot{\vec{R}}) = \sum \vec{F}_i$.

ii) For rotational motion, $\frac{d}{dt}\vec{L} = \sum \vec{r}_i \times \vec{F}_i$. [5]

where $\vec{R}$ is the position vector of the centre of mass, $\vec{L}$ is the angular momentum of the system, referred to an inertial frame of reference.

b) i) State Newton's law of restitution for a two-body collision. Show that for an elastic collision, the coefficient of restitution, e = 1.

ii) Two particles of masses m_1, m_2 and initial velocities $\vec{u}_1, \vec{u}_2$, respectively, collide head-on, if e is the coefficient of restitution. Show that the change in Kinetic energy after collision is

$\Delta T = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(1-e^2)(u_1 - u_2)^2$. [5]

7. a) A moving coordinate system Oxyz(R) has both a translational and rotational motion relative to an inertial coordinate system $O_1XYZ(S)$. The position vectors of a moving particle P are $\vec{q}$ in S, and $\vec{r}$ in R at any instant t. If $\vec{\omega}$ is the uniform angular velocity of rotation of R, show that, $\frac{d\vec{q}}{dt} = \vec{V} + \vec{u} + \vec{\omega} \times \vec{r}$, where $\vec{V}$ is the linear velocity of R w.r.t S, and $\vec{u}$ is the linear velocity of P in R. [2]

b) Hence show that the acceleration of the particle is S can be expressed as,

$\frac{d^2\vec{q}}{dt^2} = \frac{d\vec{v}}{dt} + \ddot{\vec{r}} + 2\vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$ where $\dot{\vec{r}} = \vec{u}$ and the dot$(\cdot)$ is the time-derivative in the moving frame R. Interpret the various terms on the RHS. [4]

c) A bug crawls outward from the centre with a constant speed v along the spoke of a wheel that is rotating with constant angular speed ω about a vertical axis through the centre perpendicular to the plane of the wheel.

i) Find all the forces acting on the bug.

ii) If the coefficient of friction between the spoke and the bug be μ, show that the distance the bug travels before slipping is, $d = \dfrac{(\mu^2 g^2 - 4\omega^2 v^2)^{\frac{1}{2}}}{\omega^2}$. [4]

8. a) Consider a rotating rigid body. Axis of rotation $(\hat{n})$ passes through the origin of the body reference frame OXYZ. Let $\vec{r}_i$ be the position vector of i-th particle in the body.

i) Draw a figure showing OXYZ, $\hat{n}$, $\vec{r}_i$ and write $\vec{r}_i$ in $\hat{i}, \hat{j}, \hat{k}$ basis. [2]

ii) Write $\hat{n}$ in terms of direction cosines. [1]

iii) Deduce the explicit expression for moment of inertia $I_{\hat{n}}$ about axis $\hat{n}$. [3]

b) Moment of inertia of a cube about an axis that passes through the center of mass and center of one face is I_0. Find the moment of inertia about an axis through the center of mass and one corner of the cube. (Use the explicit expression you derived in a(iii)) [4]

9. a) Consider a rigid body moves with one point stationary. Let $\vec{r}_i, \vec{v}_i$ are the radius vector and velocity respectively of the i-th particle relative to the given stationary point.

i) write the total angular momentum about that point. [1]

ii) Express $\vec{v}_i$ in terms of $\vec{\omega}$ and $\vec{r}_i$. [1]

iii) Using matrix notation find the relation between $\vec{L}$ and $\vec{\omega}$. [4]

b) Relative to certain rectangular coordinate system $OX-Y-Z$, the moments and products of inertia of a rigid body are given by $I_{xx} = 3B$, $I_{yy} = 2B$, $I_{zz} = B$, $I_{yz} = -B = I_{zy}$ and the other components are zero.

What is the kinetic energy of rotation if the body rotates about y-axis with angular velocity $\vec{\omega}$? Find also the magnitude and direction of the angular momentum. [4]

Group - C

(*Answer <u>any two</u> questions*) [2×10]

10. a) Prove that $\left(p^2 - \dfrac{E^2}{c^2}\right)$ is an invariant under Lorentz transformation. [5]

b) A particle moves in XY-plane with a velocity 6×10^9 cm/s at an angle 60° with X-axis in a system A. Find the magnitude and direction of its velocity as observed by an observer in system B, when B has a velocity 3×10^9 cm/s along the positive X-axis. [3]

c) Does the space-time diagram follow Euclidean geometry? What kind of geometry it actually follows? Explain. [2]

11. a) A spaceship is moving away from the earth at a speed v = 0·8c. When the ship is at a distance of $6{\cdot}66\times10^8$ km from the earth as measured in the earth's reference frame, a radio signal is sent out to the spaceship by an observer on earth. How long will it take for the signal to reach the ship.

i) as measured in the earth's reference frame?

ii) as measured in the ship's reference frame? Find out the location of the spaceship when the signal is received, in both frames. [7]

b) Calculate the velocity of a 1MeV electron. [3]

12. a) What is the significance of different invariant hyperbola inside the light cone in space-time diagram? [2]

b) If two events are separated by a light like intervals, is there any inertial frame in which they can be simultaneous? [2]

c) Consider two time-like (TL) four vectors $\widehat{A}$ and $\widehat{B}$ in space-time. Is there any reference frame in which $\widehat{A}\cdot\widehat{B} = 0$? Explain. [2]

d) Explain 'length contraction' and 'time dilation' in Minkowski's geometrical representation. [4]

13\. a) Write down the Lorentz transformation equations and hence derive the equations of relativistic addition of velocities. [1+5]

b) Show that two successive Lorentz transformations with velocity parameters β_1 and β_2 in the same direction are equivalent to a single Lorentz transformation of velocity parameter $\beta = \dfrac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}$. [4]

Group - D

(*Answer __any three__ questions*) [3×10]

14\. a) Establish the following relation using Fermat's principle : $\dfrac{\mu_2}{v} - \dfrac{\mu_1}{u} = \dfrac{\mu_2 - \mu_1}{r}$ (the symbols have their usual meanings). [5]

b) An achromatic converging doublet of focal length 60cm is formed with a convex lens of crown glass and a concave lens of flint glass placed in contact with each other. Calculate their focal lengths. Dispersive power of crown glass is 0·03 and that of flint glass is 0·05. [5]

15\. a) Find the cardinal points of Huygens eyepiece. What are its advantages and disadvantages with Ramsden's eye piece. [3+2]

b) The waves : $\vec{E}(z,t) = (\hat{i}E_{ox} + \hat{j}E_{oy})\cos(kz - \omega t)$, $\vec{E}'(z,t) = (\hat{i}E'_{ox} - \hat{j}E'_{oy})\cos(kz - \omega t)$ both represent a certain state of light. Show that in general such waves are not orthogonal. Under what circumstances will their planes of vibration be normal to each other. [3]

c) Describe completely the state of polorization of each of the following waves :

i) $\vec{E} = \hat{i}E_0 \sin(\omega t - kz) + \hat{j}E_0 \sin\left(\omega t - kz - \dfrac{\pi}{4}\right)$

ii) $\vec{E} = \hat{i}E_0 \cos(\omega t - kz) + \hat{j}E_0 \cos\left(\omega t - kz + \dfrac{\pi}{2}\right)$ [2]

16\. a) Distinguish between Fresnel and Fraunhofer type of diffraction. [2]

b) A parallel beam of monochromatic light is incident normally on a plane transmission grating. Find an expression for the intensity distribution of the diffracted rays due to diffraction by the grating. [6]

c) What are missing spectra in the diffraction pattern of a plane diffraction grating. [2]

17\. a) Explain the terms "Spatial Coherence" and "Temporal Coherence" with reference to Young's double slit experiment for interference pattern. [2]

b) Determine the frequency bandwidth of white light. Compute the associated coherence length and coherence time. (White light ranges from 384 THz to 769 THz) [4]

c) Derive an expression for the coherence length of a wave in terms of the linewidth $\Delta\lambda_0$ corresponding to a frequency bandwidth of $\Delta\gamma$. [4]

18\. a) Outline the theory of a zone plate and compare it with a converging lens. [6]

b) We know that principal maxima occur when $a \sin\theta_m = m\lambda \quad m = 0, \pm1, \pm2, \ldots$ and this has come to known as the grating equation for normal incidence.

i) Differentiate the grating equation (necessary for broad band of colors) [1]

ii) What is the angular separation between the sodium D lines (589·592 nm and 588·995 nm) in the first order spectrum generated by a plane transmission grating having 10,000 lines per inch at normal incidence? [3]

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